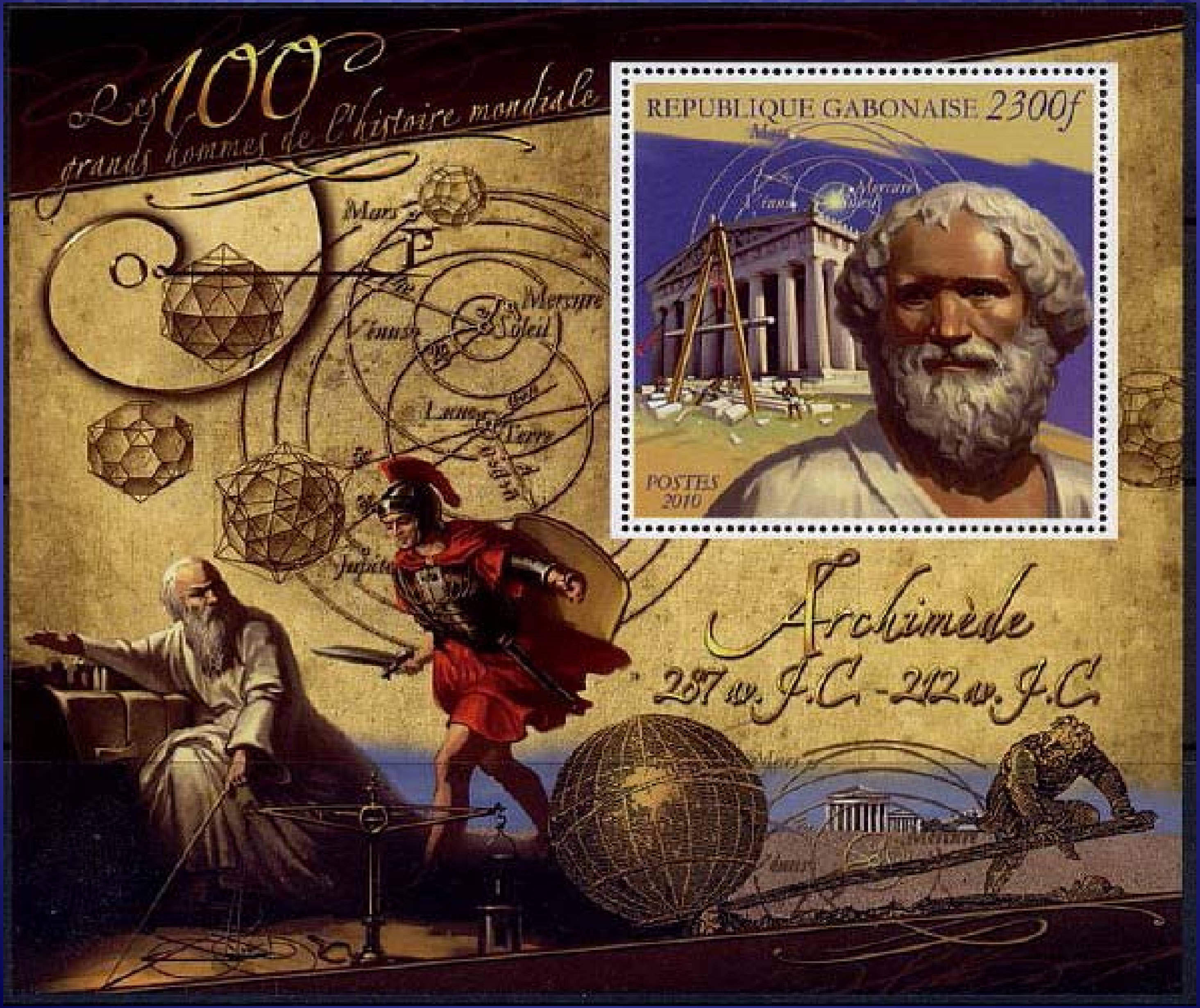
THE GENIUS OF ARCHIMEDES



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Acknowledgements

In preparing for this talk about Archimedes, the internet was very rich with resources. We want to especially note the work of Chris Rorres, *Professor Emeritus of Mathematics* Drexel University

His work:

http://www.math.nyu.edu/~crorres/Archimedes/contents.html

Digital copies of this talk can be found at:

www.MathHappy.com

Archimedes

Mathematician

Scientist

Engineer

Inventor

Author

Biography

- Born in 287 B.C., Syracuse, Sicily.
- Father, Phidias, was an astronomer.
- Studied mathematics in Alexandria, Egypt and knew Euclid’s work.
- Spent much time in creating machines used to protect Syracuse from Roman Invasion.

Inventions

Archimedes Claw

A clip!

Harry G. Harris, Professor of Structural Engineering, Drexel University

Painted by Giulio Parigi in about 1600.

Inventions

- Archimedes Death Ray
  - A clip!
  - *Cabiria*, 1914 Italian Silent Movie

Inventions

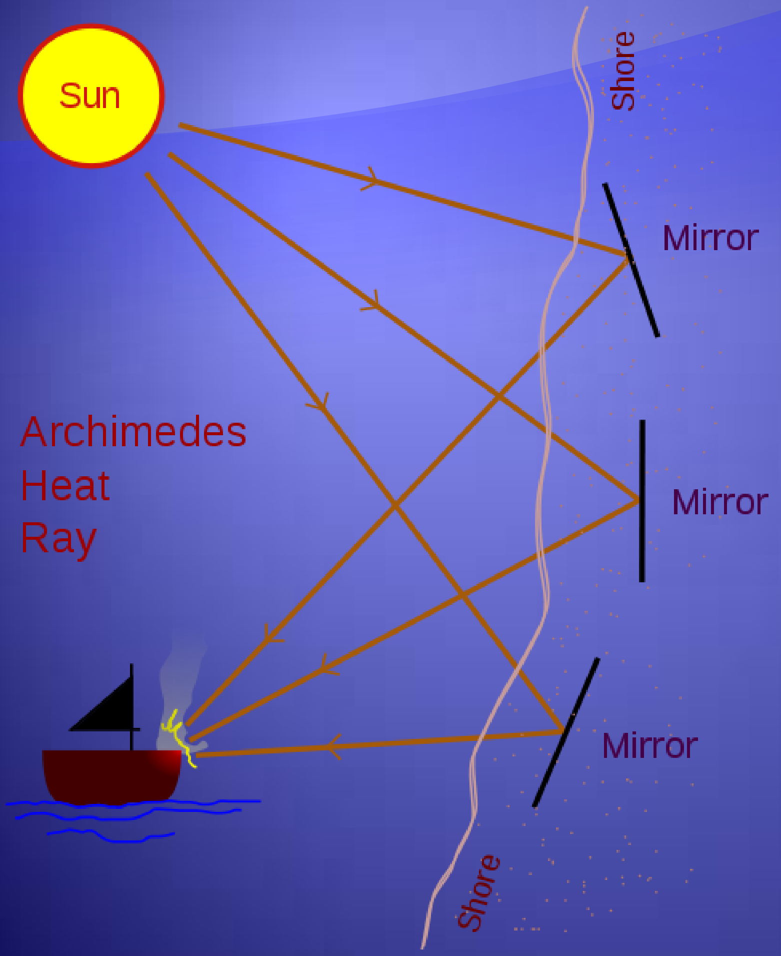


Image: WikiMedia Commons

Archimedes Death Ray

It is thought that large mirrors could be used to focus sunlight onto ships at sea.

Inventions

Archimedes Death Ray

2004 – Episode 16 *Mythbusters*

2005 – MIT Students

2006 – Episode 46 *Mythbusters* and MIT Students

Conclusions?

Death of Archimedes

A Roman invasion led to his death at the hands of a soldier in 212 B.C.

18th Century Engraving by Giovanni Maria Mazzuchelli

Number Theory

1773 Germany: Discovery of a manuscript contained a letter to Eratosthenes

"The sun god had a herd of cattle consisting of bulls and cows, one part of which was white, a second black, a third spotted, and a fourth brown. Among the bulls, the number of white ones was one half plus one third the number of the black greater than the brown; the number of the black, one quarter plus one fifth the number of the spotted greater than the brown; the number of the spotted, one sixth and one seventh the number of the white greater than the brown. Among the cows, the number of white ones was one third plus one quarter of the total black cattle; the number of the black, one quarter plus one fifth the total of the spotted cattle; the number of spotted, one fifth plus one sixth the total of the brown cattle; the number of the brown, one sixth plus one seventh the total of the white cattle. What was the composition of the herd?"

Cattle Problem

This is a system of Diophantine Equations.

W, X, Y, Z represent the number of bulls.

w, x, y, z represent the number of cows.

$$W = \frac{5}{6}X + Z$$

$$X = \frac{9}{20}Y + Z$$

$$Y = \frac{13}{42}W + Z$$

$$w = \frac{7}{12}\left(X + x\right)$$

$$x = \frac{9}{20}\left(Y + y\right)$$

$$y = \frac{11}{30}\left(Z + z\right)$$

$$z = \frac{13}{42}\left(W + w\right)$$

Cattle Problem

1880 The smallest integer solution discovered.

Bulls:

W = 10,366,482 (white)

X = 7,460,514 (black)

Y = 7,358,060 (spotted)

Z = 4,149,387 (brown)

Cows:

w =7,206,360 (white)

x = 4,893,246 (black)

y =3,515,820 (spotted)

z = 5,439,213 (brown)

There are an infinite number of solutions to the problem.

Cattle Problem

Archimedes, for fun, also required that $W + X$ be a square number and $Y + Z$ be a triangular number.

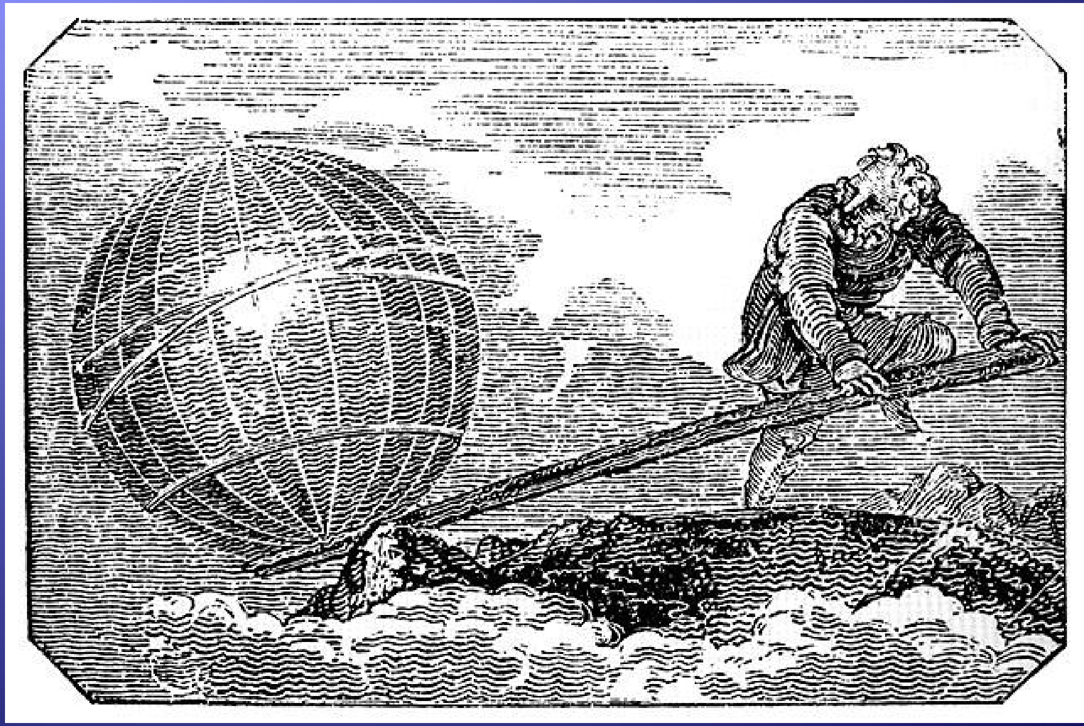
In 1965, using a computer, the solution was found. It took 42 pages to print out.

Surviving Manuscripts

On the Equilibrium of Planes
On the Measurement of a Circle
On Spirals
On the Sphere and the Cylinder
On Conoids and Spheroids
On Floating Bodies
The Quadrature of the Parabola
Ostomachion
Archimedes' Cattle Problem
The Sand Reckoner
The Method of Mechanical Theorems

On the Equilibrium of Planes

Give me a place to stand on and I will move the Earth.



On the Equilibrium of Planes

Mechanical Advantage of a lever:

$$MA = \frac{d_1}{d_2}$$

Eureka!

Hiero, King of Syracuse was concerned about crown fraud.

While bathing, Archimedes realized that the proportion of water displaced in the bath, was equal to his own volume.

A crown filled with another metal would have more volume than a crown made strictly of gold.

On Floating Bodies

Archimedes Principle

Any body completely or partially submerged in a fluid is acted upon by an upward force which is equal to the weight of the fluid displaced by the body.

Estimation for π

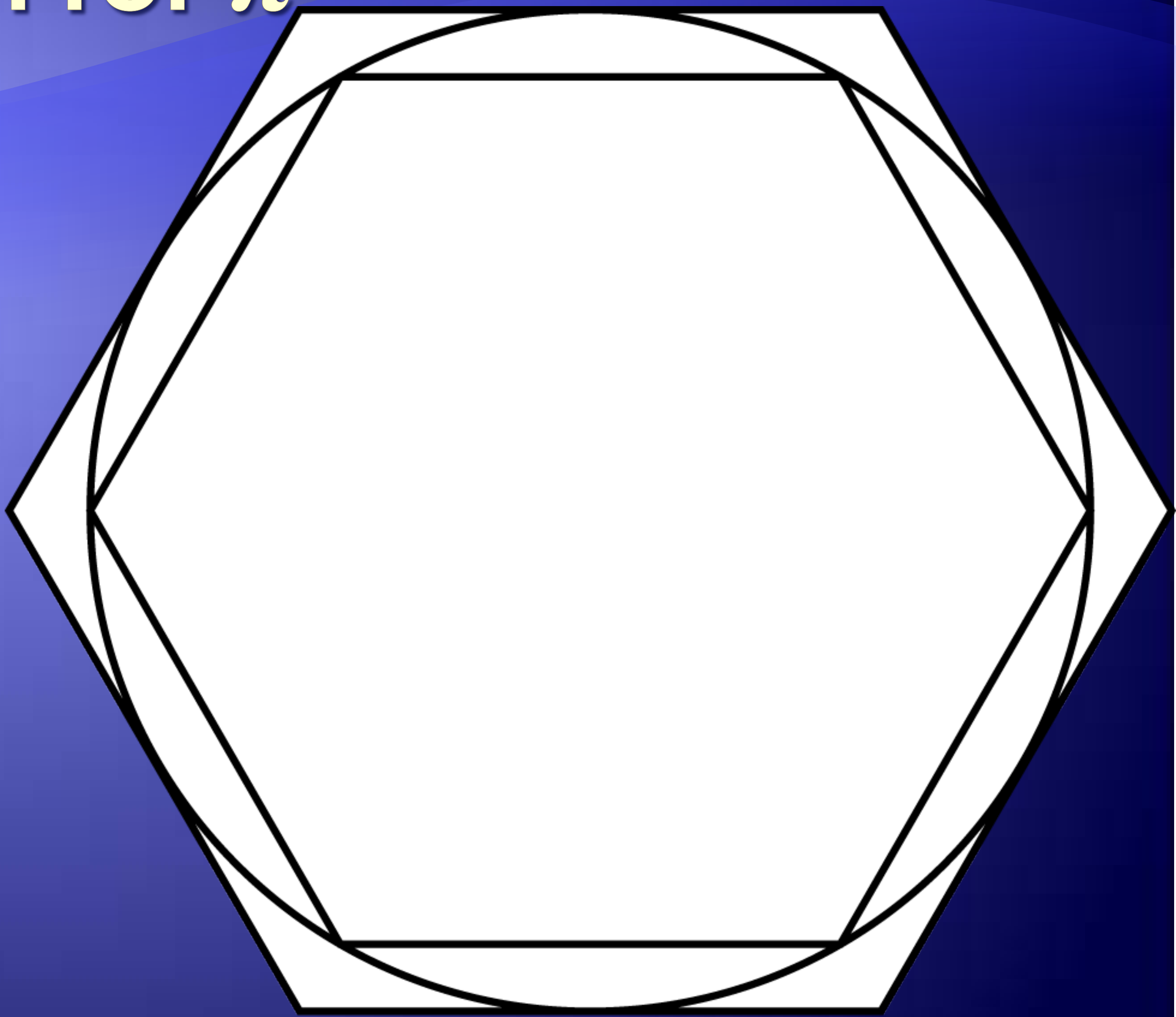
π is defined as the ratio of the circumference of a circle, to its diameter.

$$\pi = \frac{circumference}{diameter}$$

One of the first estimations for π was computed by Archimedes, in *Measurement of a Circle*.

Estimation for π

Archimedes began with a unit circle inscribed and circumscribed by regular hexagons.



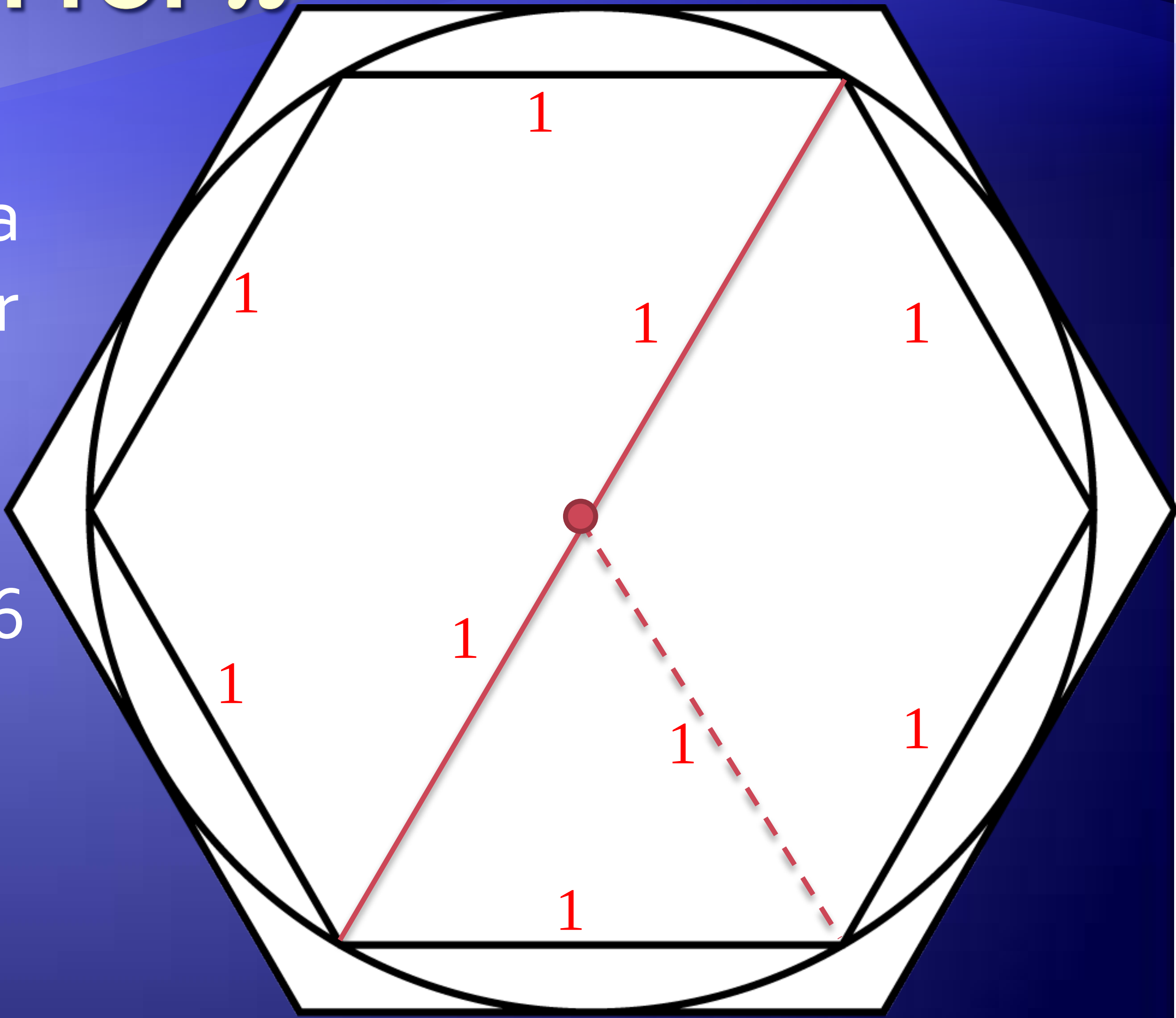
Estimation for π

The inscribed hexagon gives a lower bound for π.

Circumference = 6

Diameter = 2

$\pi = 6/2 = 3$

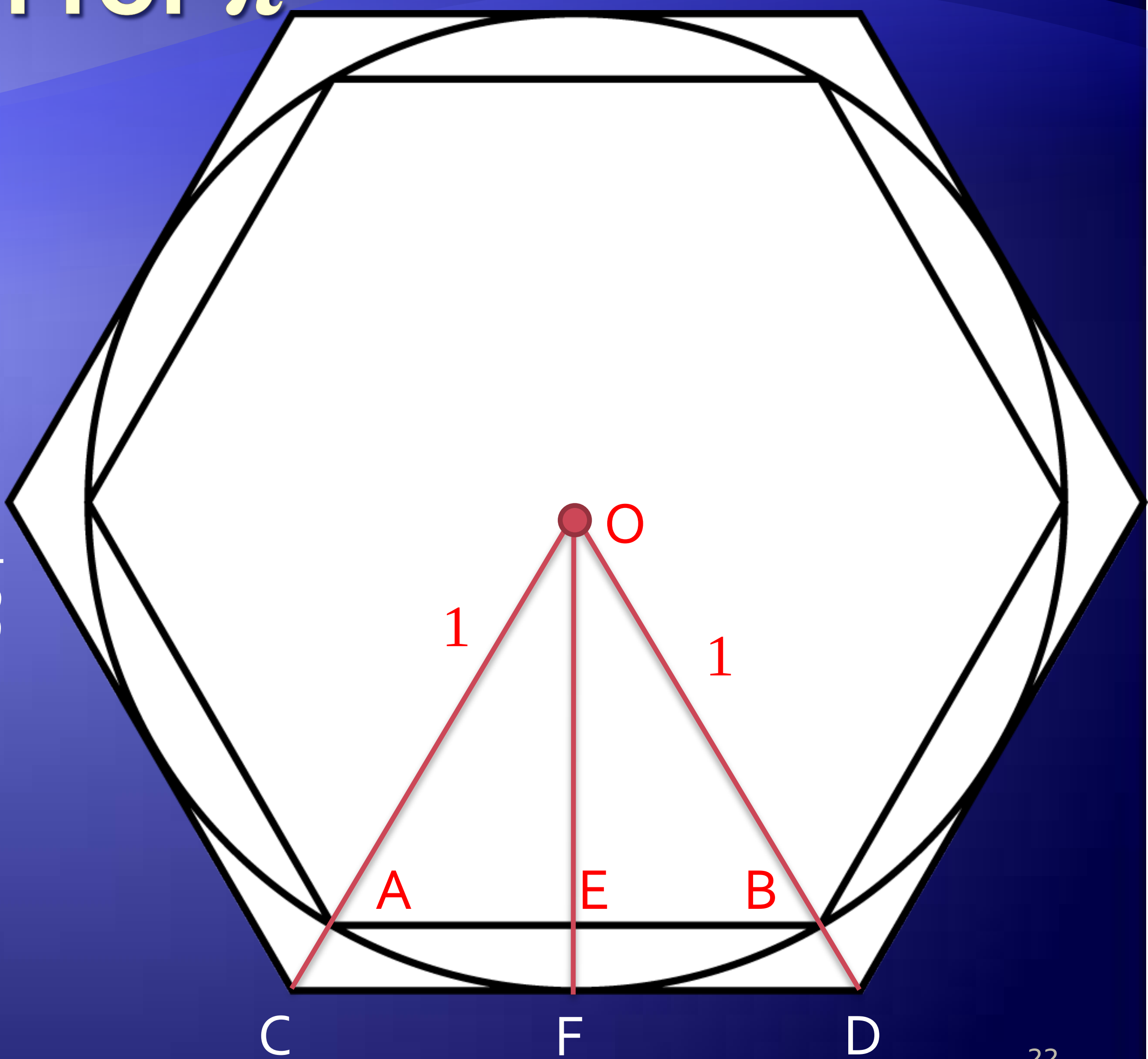


Estimation for π

- To find the upper bound, Archimedes started with an equilateral triangle.

$$\overline{AO} \cong \overline{OB} \cong \overline{AB}$$

- $\overline{OF}$ is a perpendicular bisector

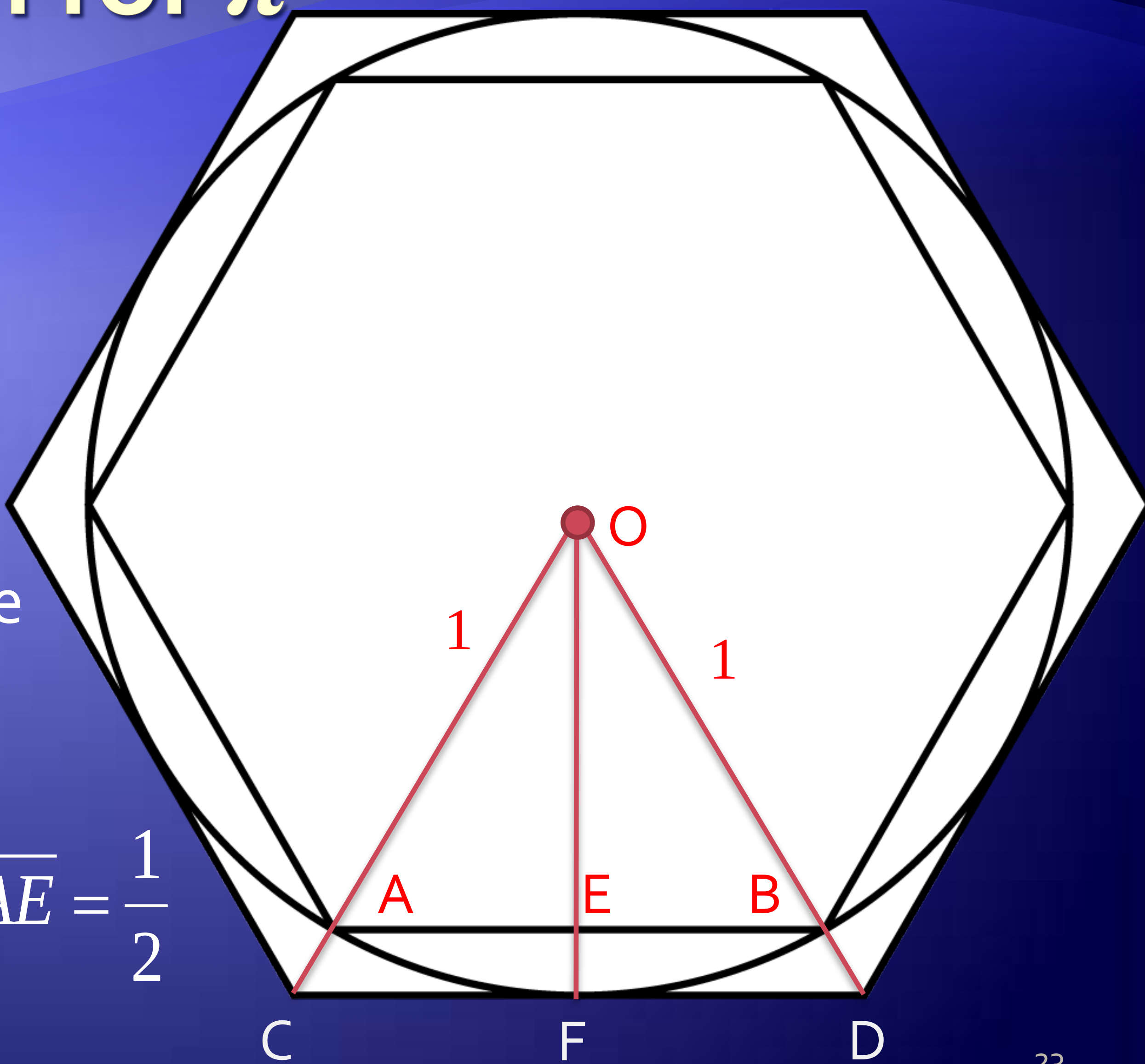


Estimation for π

ΔAOE has angles measuring 30^o, 60^o and 90^o.

The length of the legs of the triangle are known.

$$\overline{AO} = 1,\ \overline{OE} = \frac{\sqrt{3}}{2},\ \overline{AE} = \frac{1}{2}$$



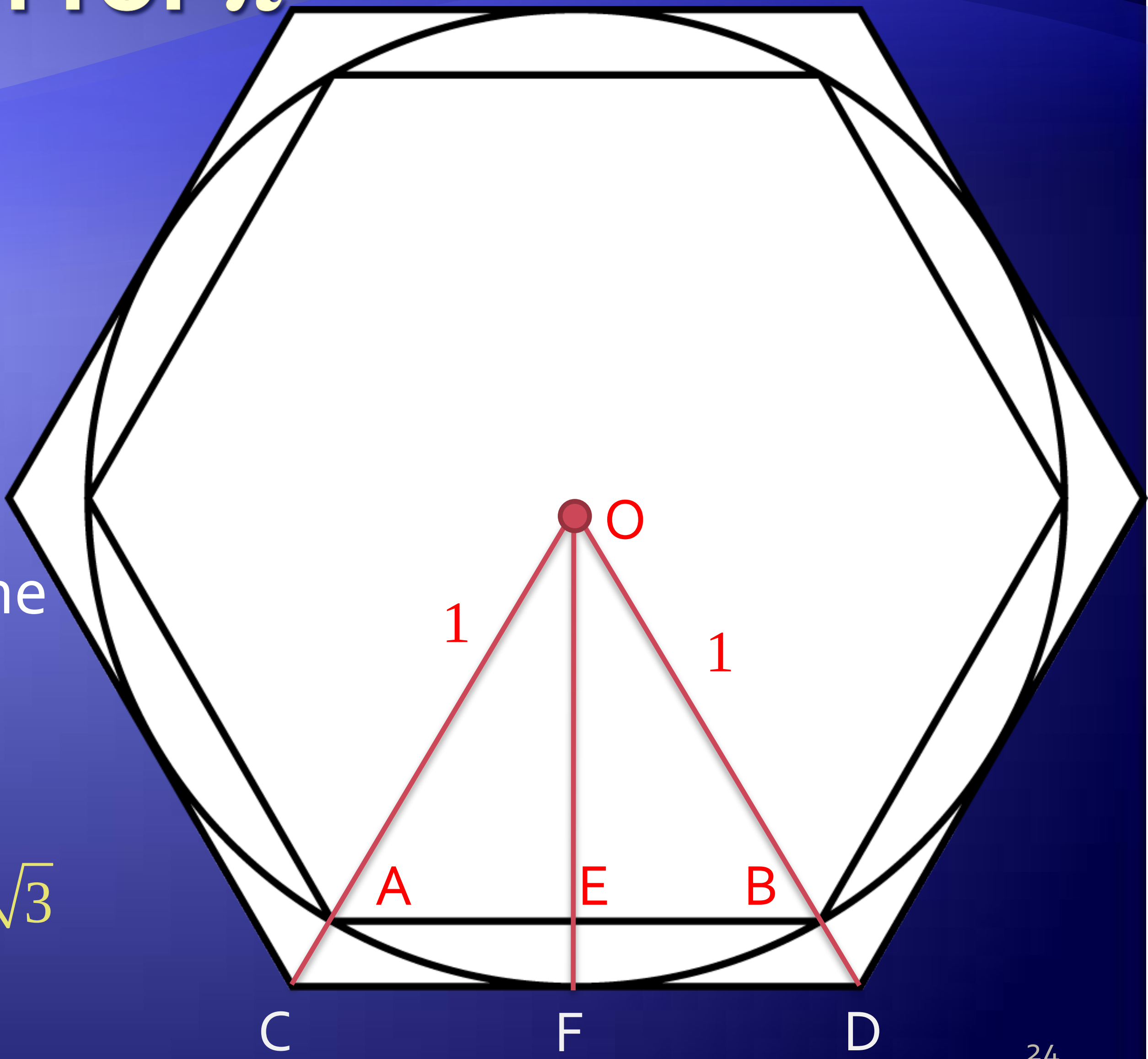
Estimation for π

Δ AOE is similar to ΔCOF

Thus: $\overline{CF} = \frac{\sqrt{3}}{3}$

The perimeter of the circumscribed hexagon:

$$12 * \overline{CF} = 12 * \frac{\sqrt{3}}{3} = 4\sqrt{3}$$



Estimation for π

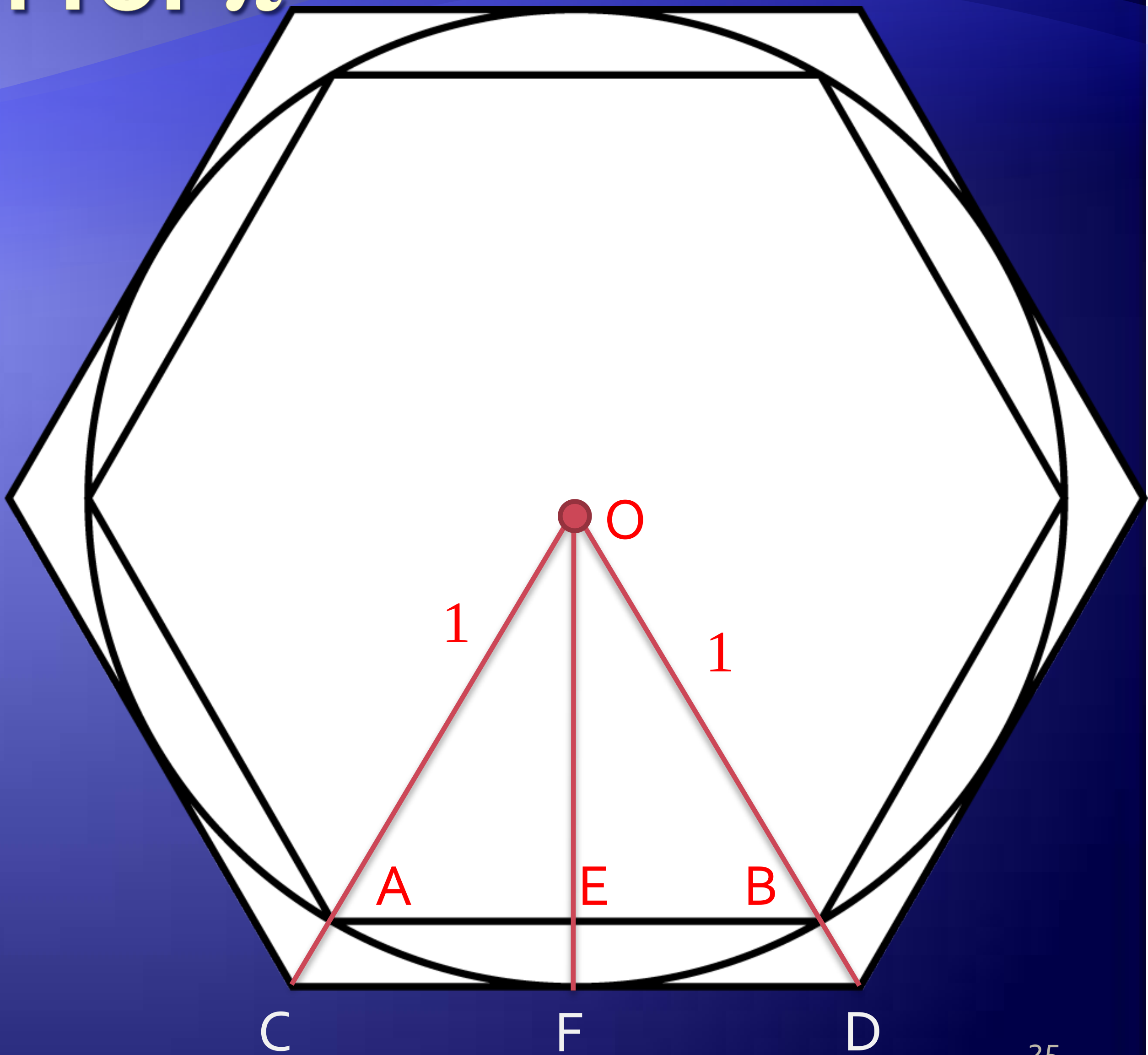
Diameter is 2.

Upper bound estimate:

$$\pi = \frac{4\sqrt{3}}{2}$$

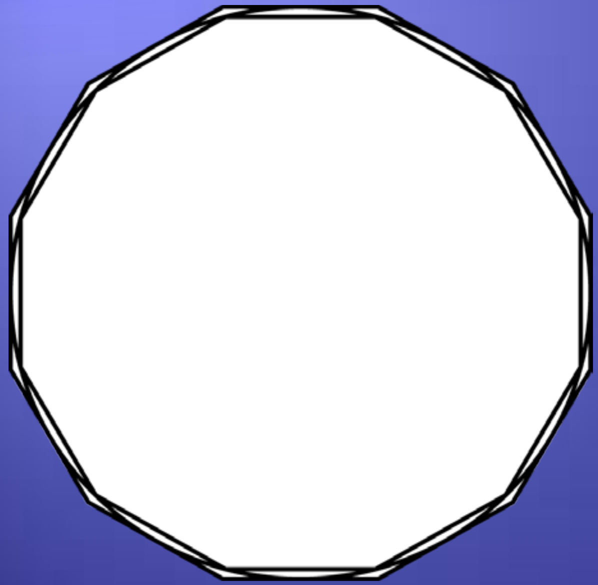
$$= 2\sqrt{3}$$

$$= 3.4641...$$

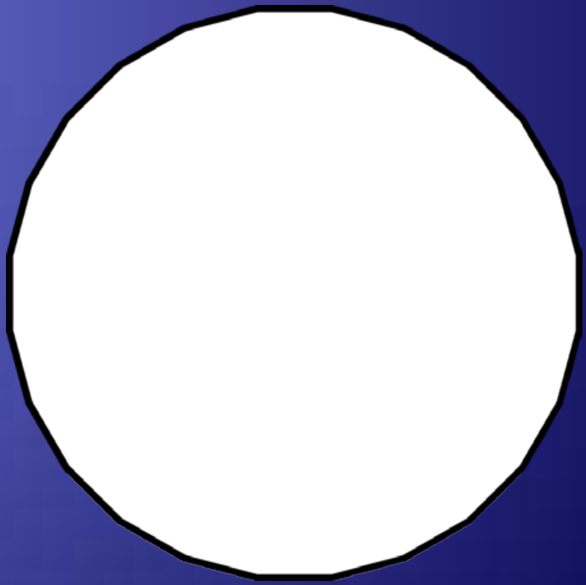


Estimation for π

Archimedes repeated this process using regular polygons, with 12, 24, 48, and 96 sides. This is a method of exhaustion.



12-gon



24-gon

Estimation for π

Archimedes final estimate:

$$3\frac{10}{71} < \pi < 3\frac{1}{7}$$

$$3.1408\ldots < \pi < 3.1429\ldots$$

Archimedes Constant!

Estimation for $\sqrt{3}$

Consider this continued fraction

$$\sqrt{3} = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \ldots}}}}}$$

Estimation for $\sqrt{3}$

Calculating this infinite fraction at finite stages yields:

$$1 = \frac{1}{1} \qquad 1 + \frac{1}{1} = \frac{2}{1} \qquad 1 + \frac{1}{1 + \frac{1}{2}} = \frac{5}{3} \qquad 1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1}}} = \frac{7}{4}$$

Each fraction is an increasingly better rational approximation for the square root of three.

Estimation for $\sqrt{3}$

$$1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\dots}}}}}$$

		1	1	2	1	2
0	1	1*1+0=1				
1	0	1*0+1=1				

Estimation for $\sqrt{3}$

$$1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\dots}}}}}$$

		1	1	2	1	2
0	1	1*1+0=1	1*1+1=2			
1	0	1*0+1=1	1*1+0=1			

Estimation for $\sqrt{3}$

$$1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\dots}}}}}$$

		1	1	2	1	2
0	1	1*1+0=1	1*1+1=2	2*2+1 =5		
1	0	1*0+1=1	1*1+0=1	2*1+1=3		

Estimation for $\sqrt{3}$

$$1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\ldots}}}}}$$

		1	1	2	1	2
0	1	1*1+0=1	1*1+1=2	2*2+1 =5	1*5+2=7	
1	0	1*0+1=1	1*1+0=1	2*1+1=3	1*3+1=4	

Estimation for $\sqrt{3}$

$$1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\ldots}}}}}$$

		1	1	2	1	2
0	1	1*1+0=1	1*1+1=2	2*2+1 =5	1*5+2=7	2*7+5 =19
1	0	1*0+1=1	1*1+0=1	2*1+1=3	1*3+1=4	2*4+3=11

Estimation for $\sqrt{3}$

$$1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\dots}}}}}$$

		1	1	2	1	2	1	2	1	2	1	2
0	1	1	2	5	7	19	26	71	97	265	362	989
1	0	1	1	3	4	11	15	41	56	153	209	571

The next entry in the table is 1351/780.

Estimation for π

Importance of this proof:

Suggest that Archimedes knew Euclid's *Elements*, which discusses in detail how to calculate the perimeter of a circumscribing polygon of $2n$ sides, given the perimeter value of a polygon with n sides.

Archimedes work demonstrated a change from previous geometric arguments into an iterative numerical process for calculating the perimeter of polygons with $2n$ sides.

The Quadrature of the Parabola

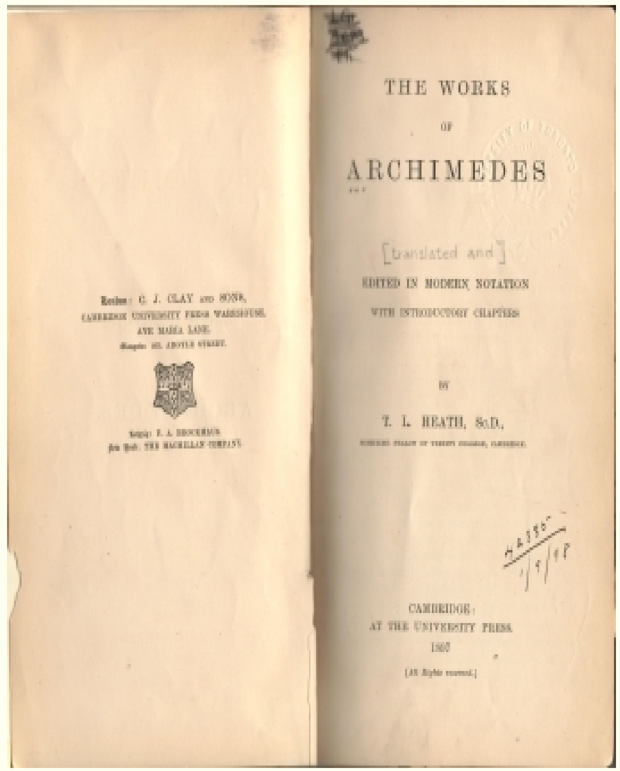
The Quadrature of the Parabola discusses 24 propositions regarding the underlying nature of parabolic segments.

Quadrature – construction of a square that has the same area of a curved shape.

Modern secondary mathematics curriculum includes discussion of lines, quadratic functions, and conic sections.

Can we somehow connect the work of Archimedes to the current secondary curriculum in a meaningful way? For students? For teachers?

The Quadrature of the Parabola



The images here have been taken from T. L. Heath's translation into English of Archimedes' collected works, published in 1897 by Cambridge University Press. Heath's edition is based in turn on the definitive Greek edition of J. L.Heiberg.

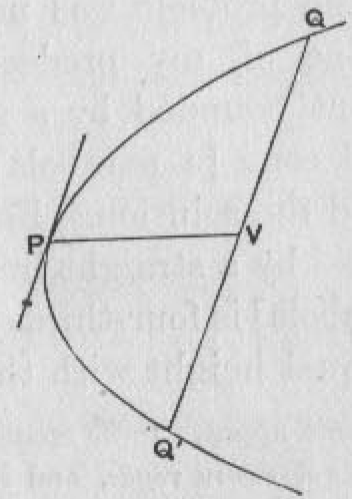
The Quadrature of the Parabola

Proposition 1.

If from a point on a parabola a straight line be drawn which is either itself the axis or parallel to the axis, as PV, and if QQ′ be a chord parallel to the tangent to the parabola at P and meeting PV in V, then

$$QV = VQ'.$$

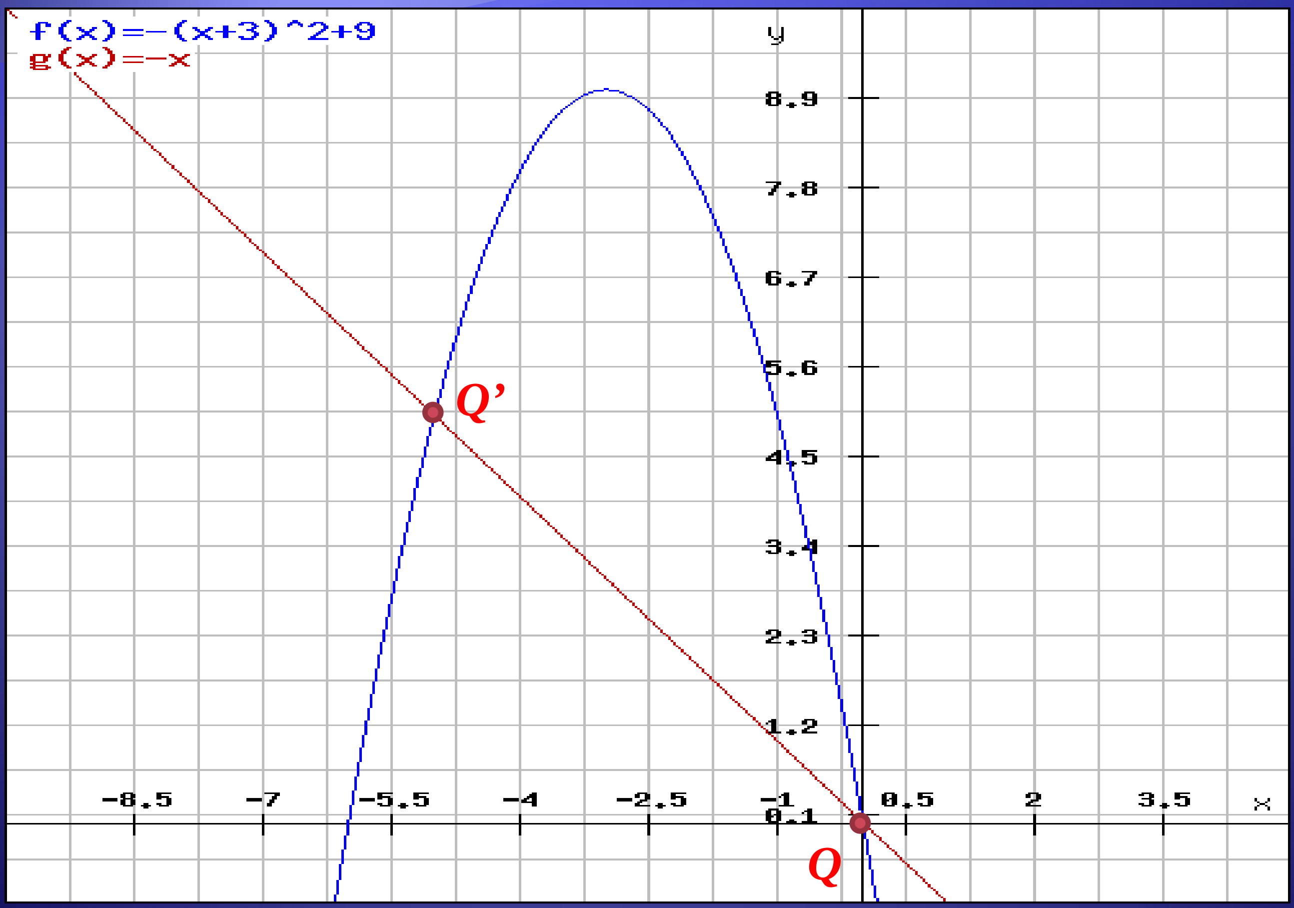
Conversely, *if* $QV = VQ'$, *the chord QQ′ will be parallel to the tangent at P.*



* The Greek of this passage is: συμβαίνει δὲ τῶν προειρημένων θεωρημάτων ἕκαστον μηδὲν ἧσσον τῶν ἄνευ τούτου τοῦ λήμματος ἀποδεδειγμένων πεπιστευκέναι. Here it would seem that πεπιστευκέναι must be wrong and that the passive should have been used.

$$f(x) = -(x+3)^2 + 9, \quad -5 \le x \le 0$$

The Quadrature of the Parabola



What do we need to get started?

The equation of Line *QQ'*

$y = -x$

$m = -1$

The Quadrature of the Parabola

Given $m = -1$, find Point P

$f'(x) = -2x - 6$

$$-2x - 6 = -1$$

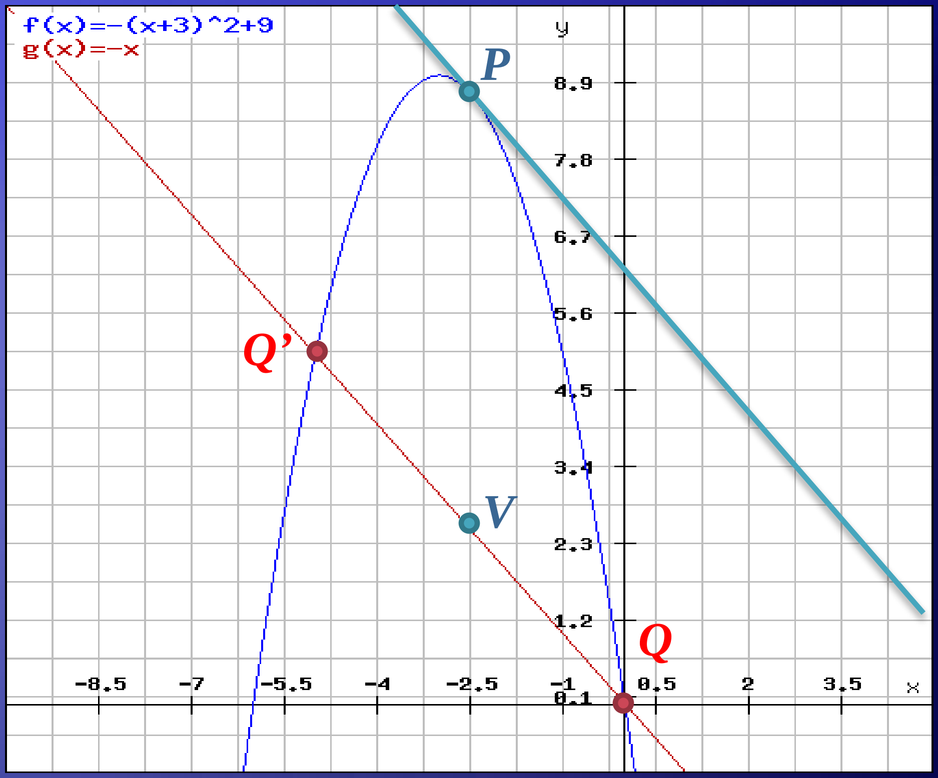
$$x = -2.5$$

This is the x-coordinate for P

$$P = (-2.5, 8.75)$$

P and V share the same x–coordinate and V is on line QQ'.

$$V = (-2.5, 2.5)$$



$$d(QV) = d(Q'V) = 2.5\sqrt{2}$$

The Quadrature of the Parabola

Topics Covered in this one exercise:

- Given two points find a line
- Parallel and Tangent Lines
- Graphing functions
- Function Evaluation
- Distance between two points
- Derivative of a function
- Derivative input for a specific slope value.

The Quadrature of the Parabola

Outcomes:

Connection between current topics of modern curriculum that are often appreciated disparately.

Connection between current mathematics curriculum to mathematics history.

Allow for incorporation of Core State Standards of problem solving, communication.

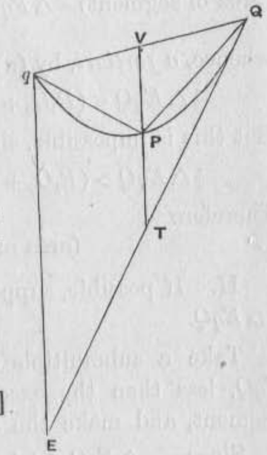
Proposition 17

The area of any segment of a parabola is four-thirds of the triangle which has the same base as the segment and equal height.

Proposition 17.

It is now manifest that *the area of any segment of a parabola is four-thirds of the triangle which has the same base as the segment and equal height.*

Let Qq be the base of the segment, P its vertex. Then PQq is the inscribed triangle with the same base as the segment and equal height.



Since P is the vertex* of the segment, the diameter through P bisects Qq. Let V be the point of bisection.

Let VP, and qE drawn parallel to it, meet the tangent at Q in T, E respectively.

Then, by parallels,

$$qE = 2VT,$$

and $$PV = PT, \quad \text{[Prop. 2]}$$

so that $$VT = 2PV.$$

Hence $$\triangle EqQ = 4\triangle PQq.$$

But, by Prop. 16, the area of the segment is equal to $\frac{1}{3}\triangle EqQ$.

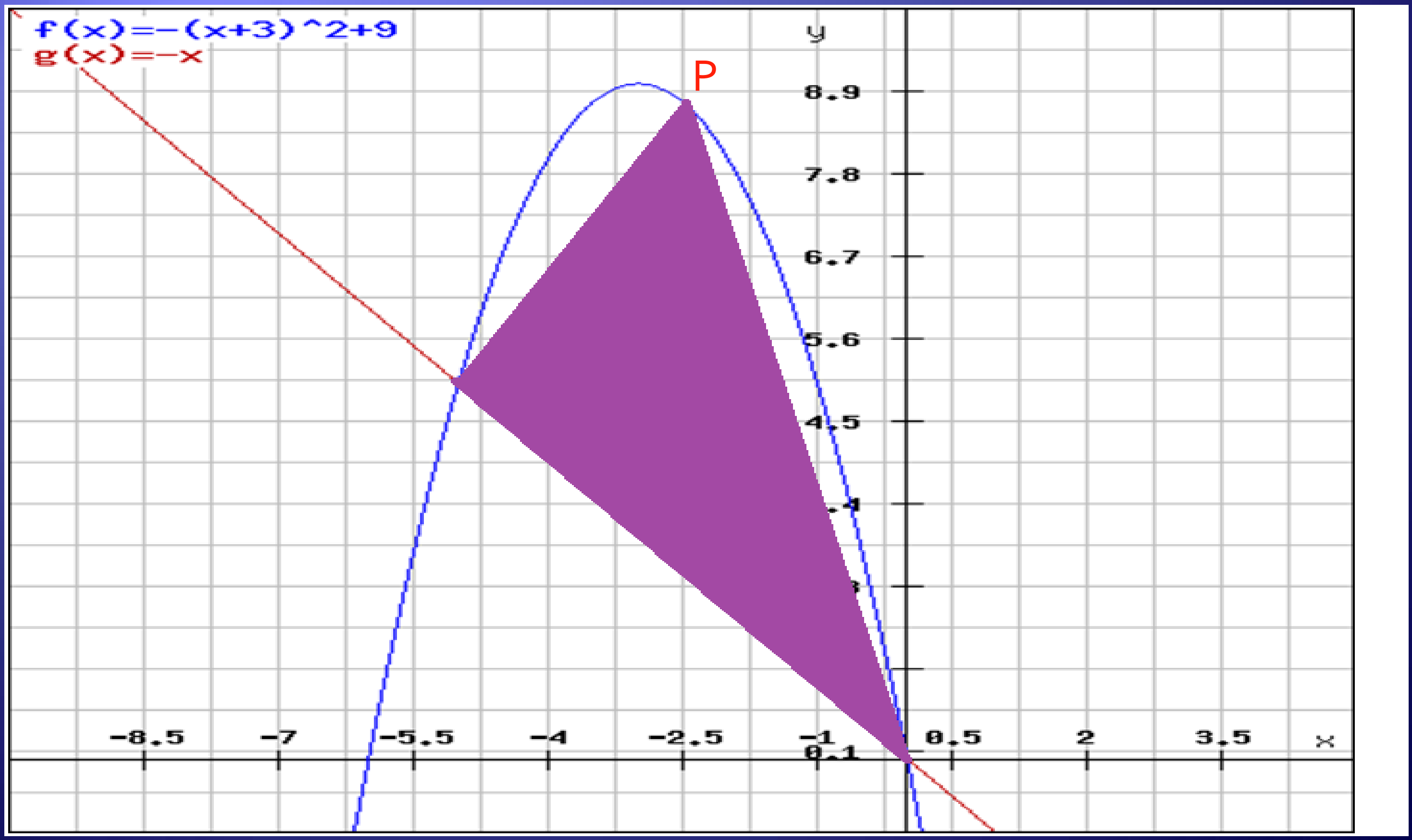
Therefore $$(\text{area of segment}) = \tfrac{4}{3}\triangle PQq.$$

Def. "In segments bounded by a straight line and any curve I call the straight line the **base**, and the **height** the greatest perpendicular drawn from the curve to the base of the segment, and the **vertex** the point from which the greatest perpendicular is drawn."

* It is curious that Archimedes uses the terms *base* and *vertex* of a segment here, but gives the definition of them later (at the end of the proposition). Moreover he assumes the converse of the property proved in Prop. 18.

Proposition 17

The area between the parabola and the base is 4/3 the area of the inscribed triangle.



Proposition 17

A modern approach to find the area between the parabola and the base uses integration.

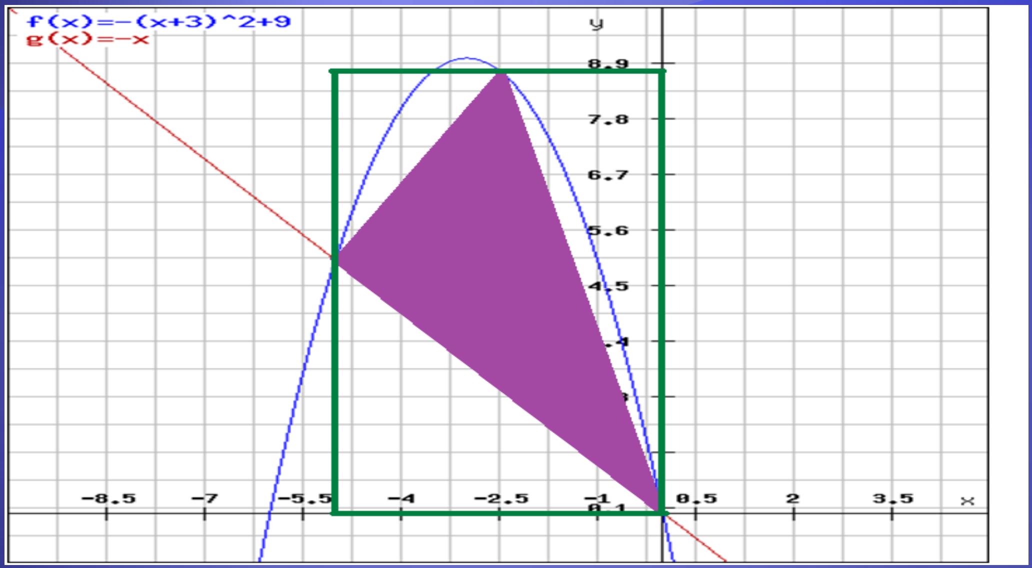
$$\int_{-5}^{0} (-x^2 - 6x) - (-x)dx$$

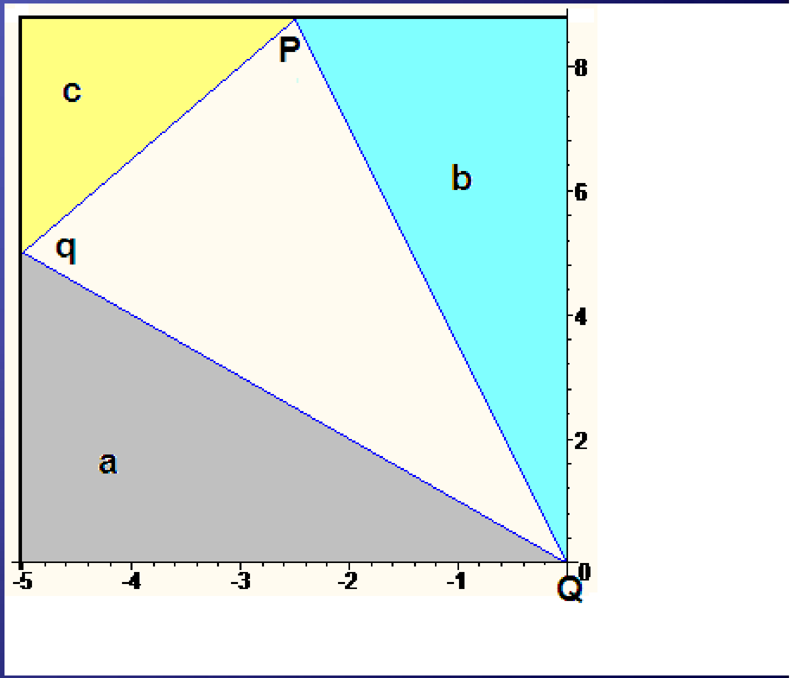
$$= \int_{-5}^{0} (-x^2 - 5x)dx$$

$$= \frac{-x^3}{3} - \frac{5x^2}{2}\Bigg|_{-5}^{0}$$

$$= \frac{125}{6}$$

Proposition 17





Proposition 17

Area of Rectangle:

5 * (8.75) = 43.75

Area of section a:

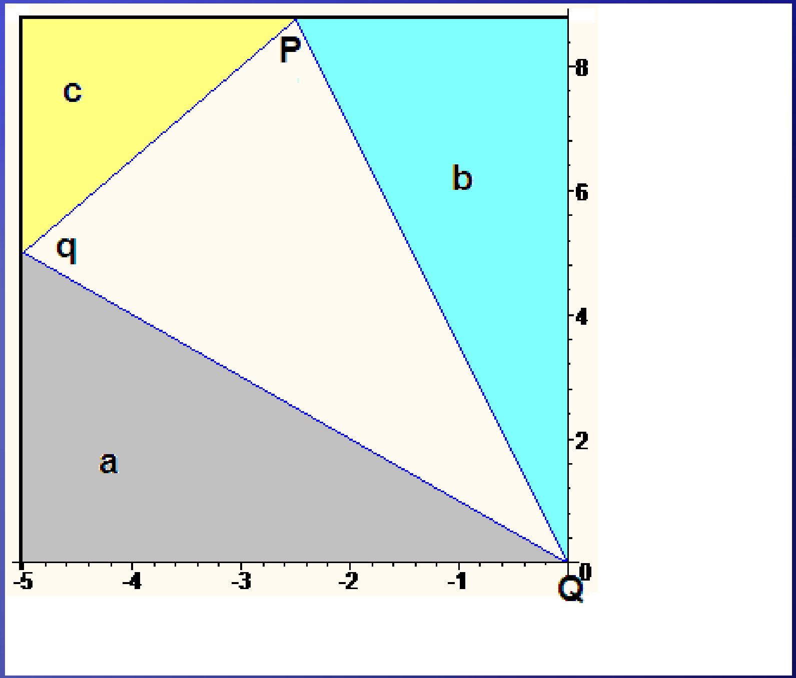
0.5*5*5 = 12.5

Area of section b:

0.5*2.5*8.75 = 10.9375

Area of section c:

0.5*2.5*3.75 = 4.6875

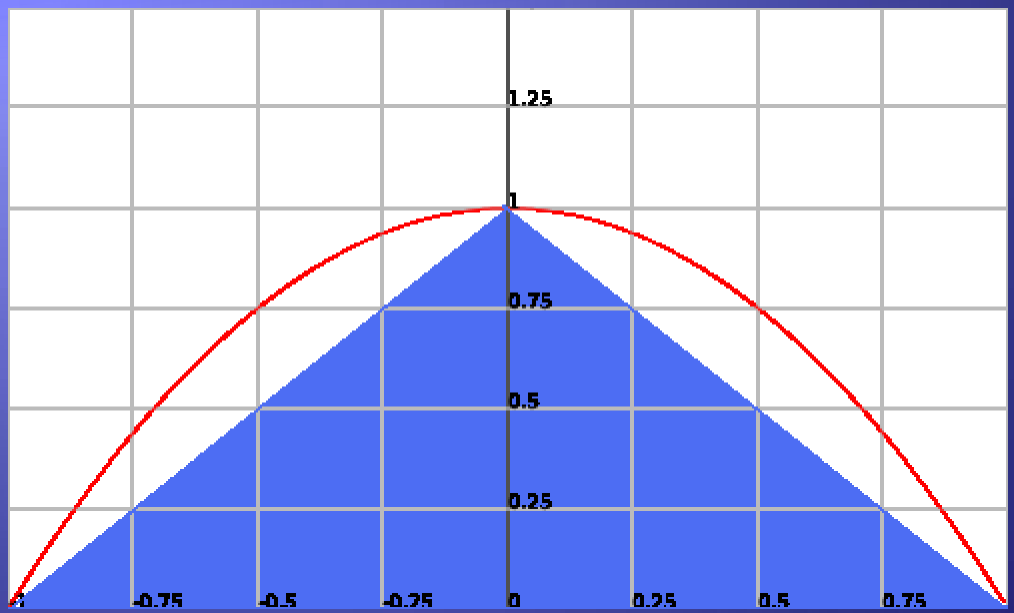


Area = 125/8

4/3 Area = 125/6

Consider a Geometric Argument

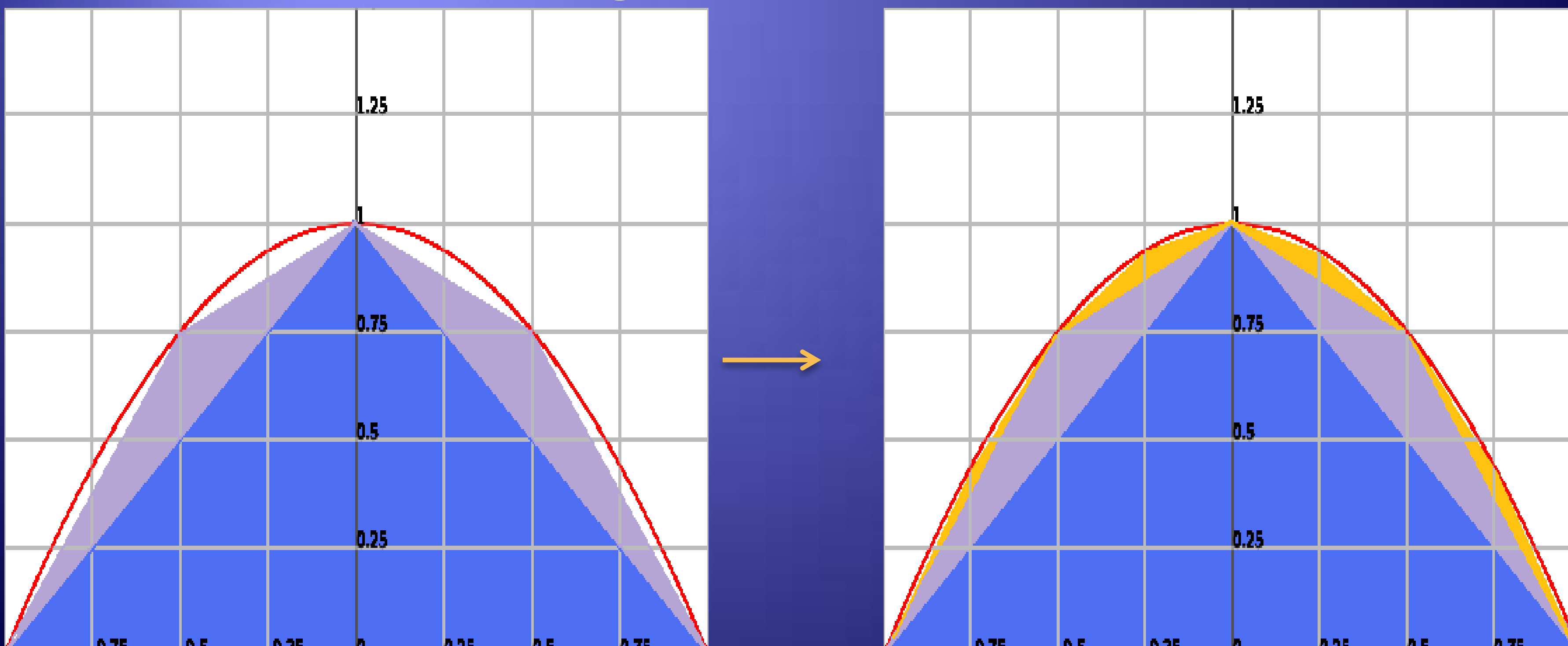
Archimedes would have started with a parabolic segment and corresponding triangle.



$$f(x) = 1 - x^2$$

Consider a Geometric Argument

- Next he constructed two smaller triangles, whose total area equals ¼ of the area of the previously constructed triangle.



Proposition 23

Given a series of areas A, B, C, D, …,Z, of which A is the greatest, and each of which is equal to four times the next in order, then

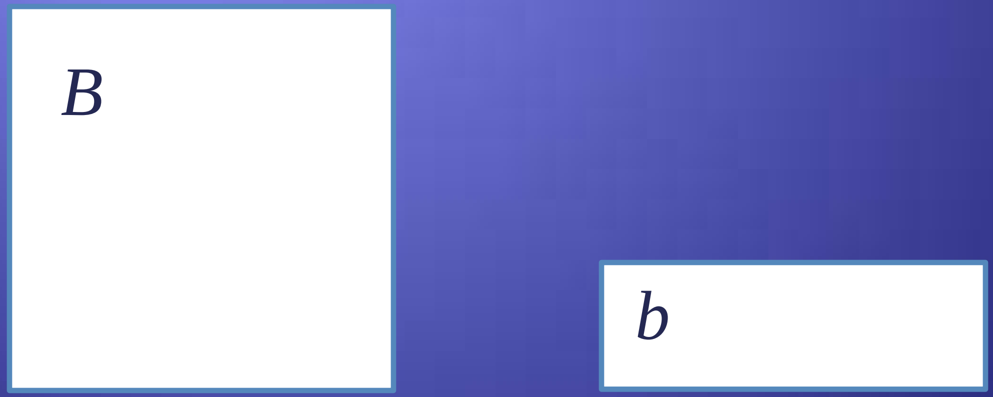
$$A + B + C + \ldots + Z + (1/3)Z = (4/3)\,A$$

Proposition 23

Areas are defined (and drawn)

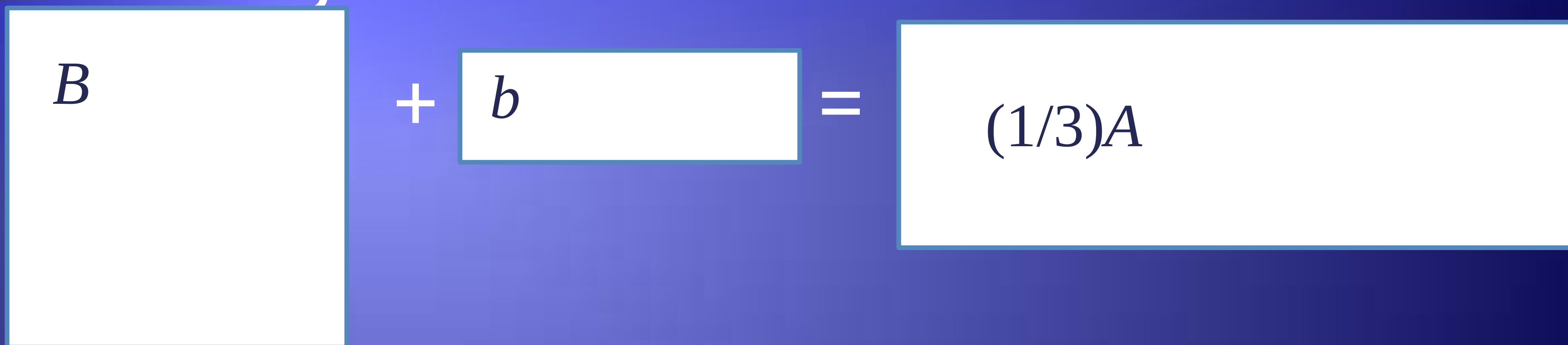
$b = (1/3)B$

$c = (1/3)C$ and so on …



Proposition 23

Adding the areas he finds:



Given $4B = A$:

$B + b$

$= (1/4)\,A + (1/3)(1/4)\,A$

$= (1/3)\,A$ and so on.....

Proposition 23

Algebraically we can sum all areas:

$$B + C + D + \ldots + Z + b + c + d + \ldots + z = (1/3)\,(A + B + C + D + \ldots + Y)$$

and

$$b + c + d + \ldots + y = (1/3)\,(B + C + D + \ldots + Y)$$

Subtracting yields

$$B + C + D + \ldots + Z + z = (1/3)\,A$$

Thus:

$$A + B + C + D + \ldots + Z + z = (4/3)\,(A)$$

Proposition 23

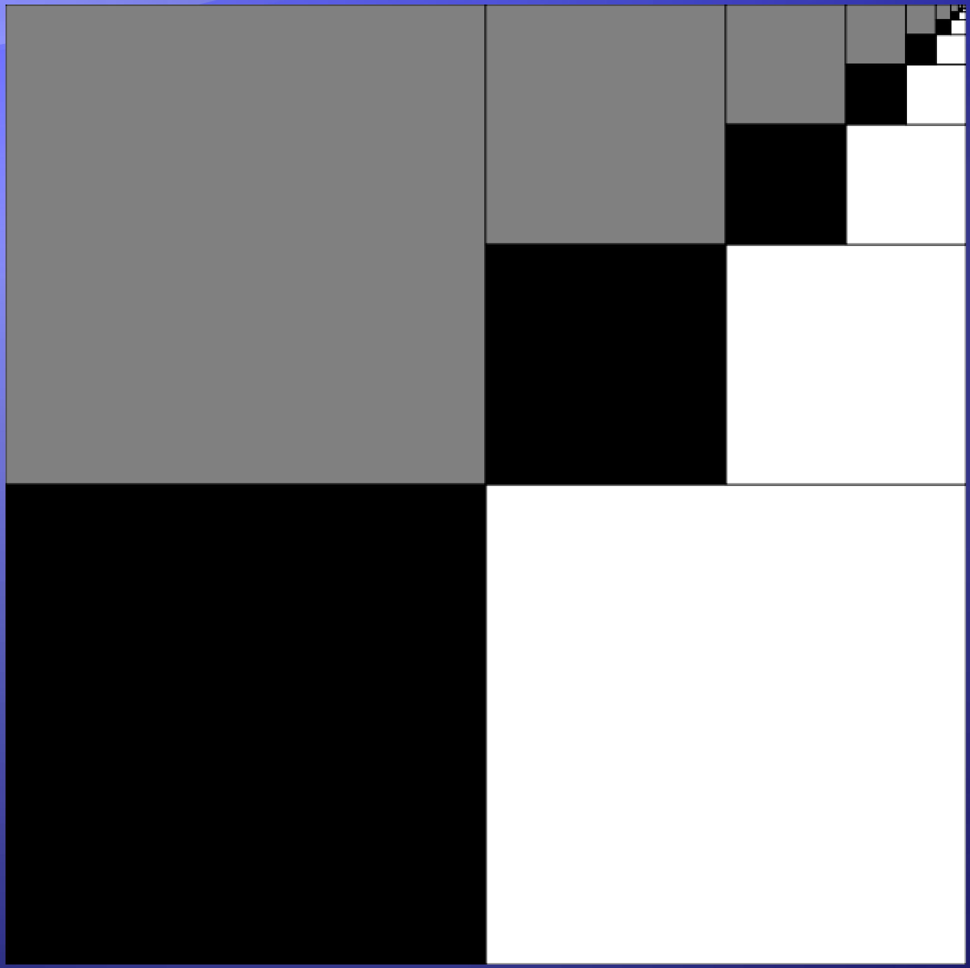


Image: Wikimedia Commons

Proposition 23

This is the algebraic equivalent to:

$$1+\frac{1}{4}+\left(\frac{1}{4}\right)^{2}+\ldots+\left(\frac{1}{4}\right)^{n-1}$$

$$=\frac{4}{3}-\frac{1}{3}\left(\frac{1}{4}\right)^{n-1}$$

$$=\frac{1-\left(\frac{1}{4}\right)^{n}}{1-\frac{1}{4}}$$

Proposition 23

Further observations:

- Archimedes argues that in general when considered two entities X and Y, either $X < Y$, $X = Y$ or $X > Y$.

- If you can show that $X < Y$ or $X > Y$ cannot happen, then X must equal Y.

- Archimedes applies these observations to the problem of the relationship between the area of a parabolic segment and the related triangle.

Archimedes *Palimpsest*

Archimedes *Palimpsest*

820 Leo the Geometer had copies of Archimedes's work in Constantinople.

1229 Religious scribes reused ancient texts to create prayer books. Eventually moved to Jerusalem.

1840 moved back to Constantinople.

1876 One folio sheet found in Cambridge University Library.

1907 discovered in Constantinople, by John Ludwig Heiberg, leading Archimedean scholar. The manuscript disappeared during the First World War.

1999 The *Palimpsest* arrives *at* Walters Museum in Baltimore.

Archimedes *Palimpsest*

Damaged manuscript, mold, ink and paint

Missing Pages; Painted Images

High tech computer photography work, e.g., Photoshop and spectrum Analysis

Community of scholars formed: mathematicians, imagers, physicists, linguists, archival experts, restoration experts

Archimedes *Palimpsest*

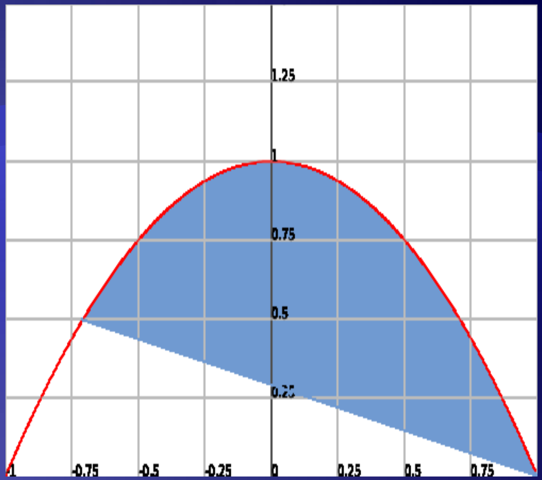
- Potential Infinity
  - Very large (or small) but essentially finite values, chosen for convenience.

- Actual Infinity
  - The number of points are on a line.

- Proofs done by Ancient Greek Mathematicians made use of potential infinity, but avoided use of actual infinity.

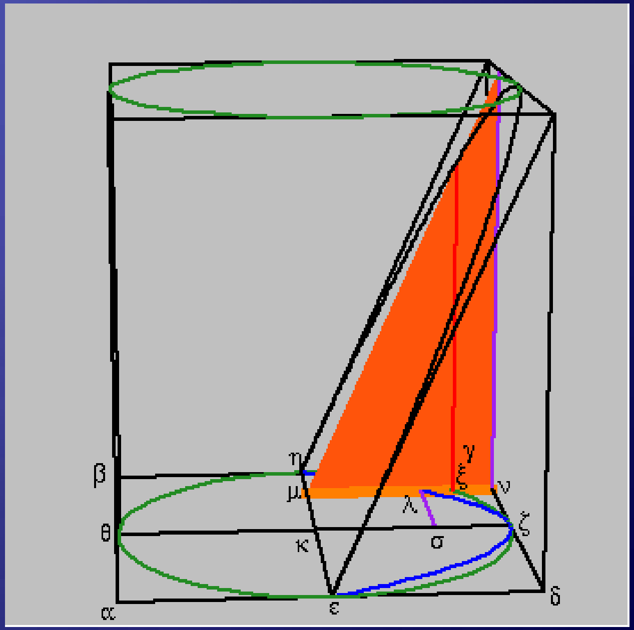
The Method

In proposition 1: Archimedes calculates the area of a parabolic segment. His work reflects what we know as modern calculus.



In the 17th century, mathematicians Isaac Newton and Gottfried Leibnitz created integral calculus.

In proposition 14 volume of a cylinder calculated using slices by an oblique plane. His proof makes an infinite number of triangles.



This is a reference to actual infinity. This discovery was made in 2001.

Stomachion

Archimedes Box

14 piece puzzle that when assembled makes a square.

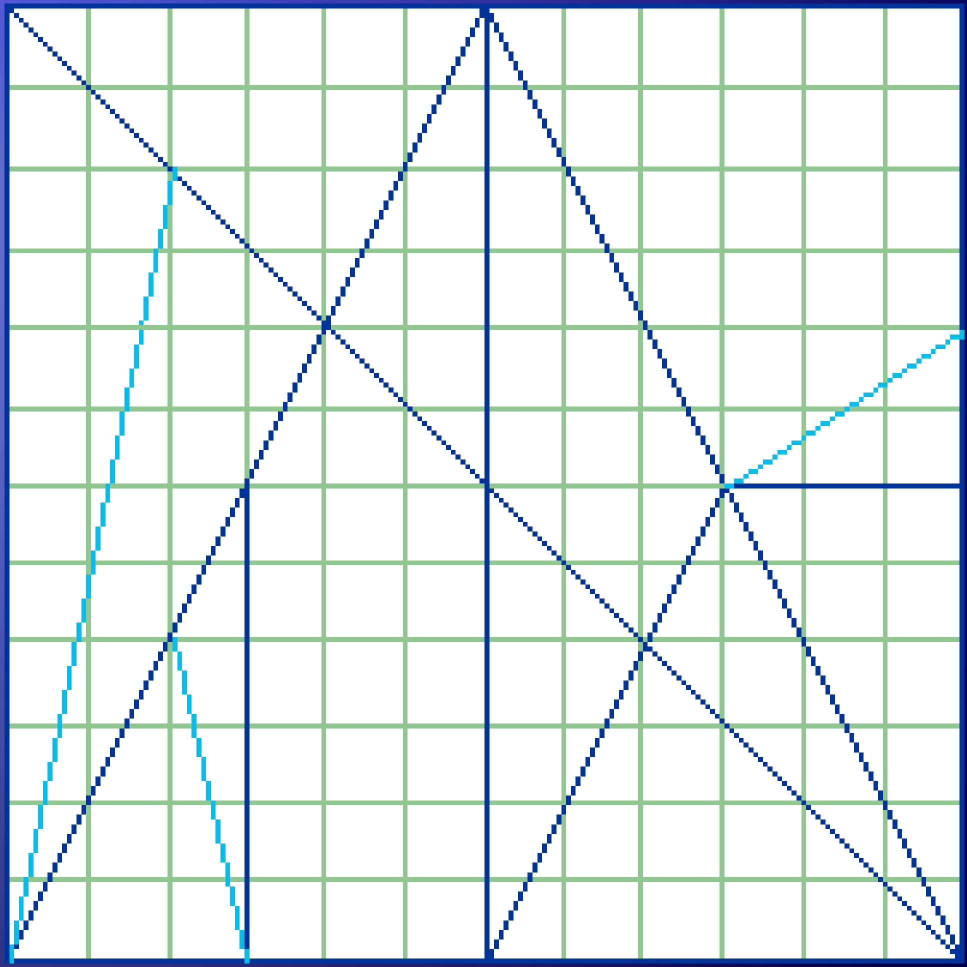
There are 536 distinct solutions.

Allowing for rotations and reflections, the number grows to 17,152.

This was proven in 2003.

Stomachion

- This images shows the piece shapes.
- The grid is 12 x 12.
- The vertices of each piece lie on a vertex, making lattice polygons.



Pick's Theorem

Given:

L = number of lattices points internal to the lattice polygon

B = number of lattice points on the boundary of the lattice polygon

Then area of lattice polygon:

$$\text{Area} = L + B/2 - 1$$

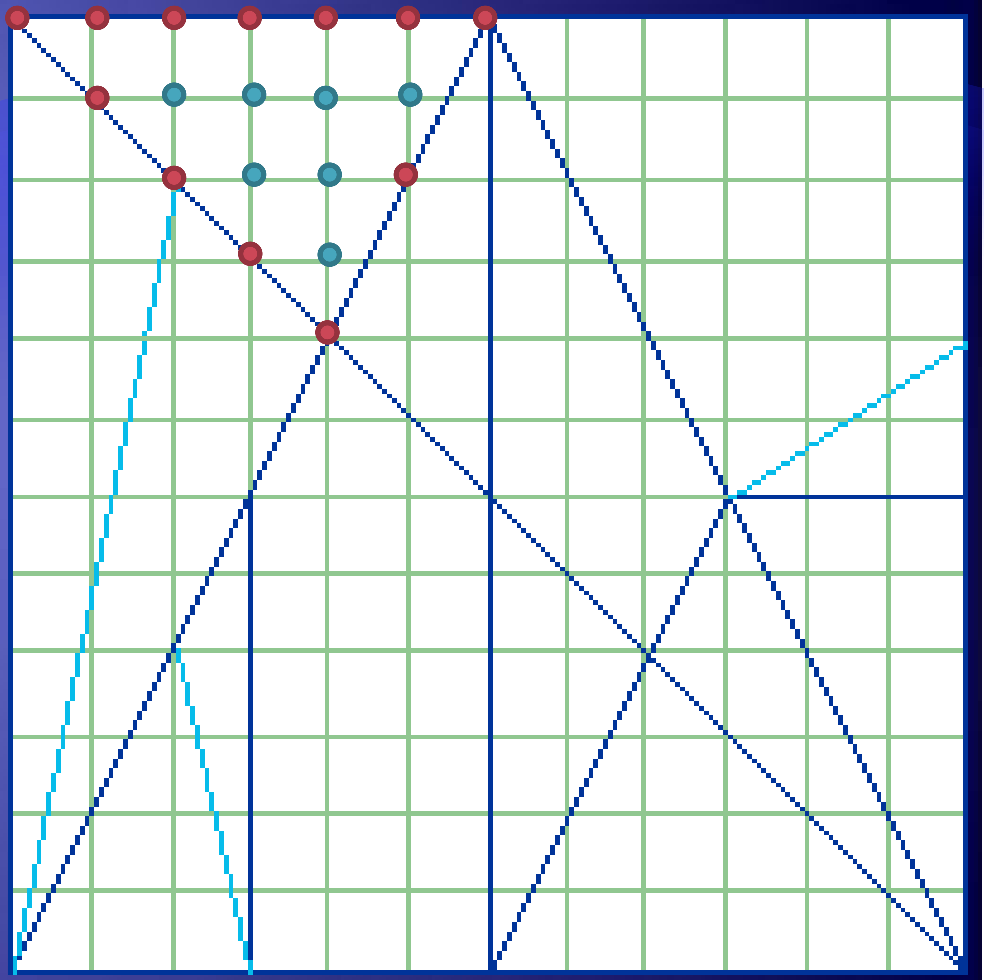
Pick's Theorem

L points are noted in blue.

B points are noted in red.

Area = $L + B/2 - 1$

Area $= 7 + 12/2 - 1 = 12$



- Stop!

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